

Boundary perturbations due to the presence of small linear cracks in an elastic body*

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Abstract

In this paper, Neumann cracks in elastic bodies are considered. We establish a rigorous asymptotic expansion for the boundary perturbations of the displacement (and traction) vectors that are due to the presence of a small elastic linear crack. The formula reveals that the leading order term is ε^2 where ε is the length of the crack, and the ε^3 -term vanishes. We obtain an asymptotic expansion of the elastic potential energy as an immediate consequence of the boundary perturbation formula. The derivation is based on layer potential techniques. It is expected that the formula would lead to very effective direct approaches for locating a collection of small elastic cracks and estimating their sizes and orientations.

Mathematics subject classification (MSC2000): 35B30, 74B05

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1 Introduction

The displacement (or traction) vector can be perturbed due to the presence of a small crack in an elastic medium. The aim of this paper is to derive an asymptotic formula for the boundary perturbations of the displacement as the length of the crack tends to zero. The focus is on cracks with homogeneous Neumann boundary conditions, *i.e.*, perfectly insulating cracks. We consider the linear isotropic elasticity system in two dimensions and assume that the crack is a line segment of small size. The derivation of the asymptotic formula is based on layer potential techniques.

The paper extends recent asymptotic results that have been used for an efficient imaging of small defects. In [8] an electrostatic model, where the crack is perfectly conducting, was considered and an asymptotic expansion of the boundary perturbations that are due to the presence of a small linear crack was derived. The asymptotic formula leads us to efficient algorithms to detect cracks using boundary measurements [3, 8]. Their resolution and stability of the algorithms with respect to medium and measurement noises

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were investigated in [2, 4]. There were some work on boundary perturbation due to the presence of small inclusions in linear elasticity; the effect of small inclusions on boundary measurements has been studied in [5, 9]. The effect of thin elastic inclusions on boundary measurements was quantified in [10, 12]. Direct reconstruction algorithms for locating small or thin elastic defects were developed in [1, 11, 15, 16]. We emphasize that the results of this paper (on cracks) can *not* be obtained as a limiting case of thin inclusions.

The results of this paper reveals that the leading order term of the boundary perturbation is ε^2 where ε is the length of the crack, and its intensity is given by the traction force of the background solution on the crack (see Theorem 4.1.) We also prove that the ε^3 -order term vanishes. By integrating the boundary perturbation formula against the given traction, we are able to derive an asymptotic expansion for the perturbation of the elastic potential energy, which is an improvement over already existing results [26, 22] (see the discussion at the end of Section 5).

The boundary perturbation formula derived in this paper carries information about the location, size, and orientation of the crack, and we expect, as in the electrostatic case, that the formula will provide a powerful tool to solve the inverse problem of identifying the cracks in terms of boundary measurements. The implementation of imaging algorithms based on the present expansion and the analysis of their resolution and stability will be the subject of a forthcoming paper.

The paper is organized as follows. In section 2, a representation formula for the solution of the problem in the presence of a Neumann crack is derived. Section 3 is devoted to making explicit the hyper-singular character involved in the representation formula. Using analytical results for the finite Hilbert transform, we derive in section 4 an asymptotic expansion of the effect of a small Neumann crack on the boundary values of the solution. Section 5 aims to derive the topological derivative of the elastic potential energy functional. Appendix contains technical calculation of the double layer potential.

2 A representation formula

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain, whose boundary $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. We assume that Ω is a homogeneous isotropic elastic body so that its elasticity tensor $\mathbb{C} = (C_{ijkl})$ is given by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (2.1)$$

with the Lamé coefficients λ and μ satisfying $\mu > 0$ and $\lambda + \mu > 0$. Let $\gamma_\varepsilon \subset \Omega$ be a small straight crack with size ε , located at some fixed distance d_0 from $\partial\Omega$, *i.e.*,

$$\text{dist}(\gamma_\varepsilon, \partial\Omega) \geq d_0.$$

We denote by \mathbf{e}^\perp a unit normal to γ_ε .

Let

$$\Psi := \left\{ \psi : \partial_i \psi_j + \partial_j \psi_i = 0, \quad 1 \leq i, j \leq 2 \right\},$$

or equivalently,

$$\Psi = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} y \\ -x \end{bmatrix} \right\}.$$

Introduce the space

$$L_{\Psi}^2(\partial\Omega) := \left\{ \mathbf{f} \in L^2(\partial\Omega) : \int_{\partial\Omega} \mathbf{f} \cdot \boldsymbol{\psi} \, d\sigma = 0 \quad \text{for all } \boldsymbol{\psi} \in \Psi \right\}.$$

Let \mathbf{u}_{ε} be the displacement vector caused by the traction $\mathbf{g} \in L_{\Psi}^2(\partial\Omega)$ applied on the boundary $\partial\Omega$ in the presence of γ_{ε} . Then \mathbf{u}_{ε} is the solution to

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_{\varepsilon}) = 0 & \text{in } \Omega \setminus \bar{\gamma}_{\varepsilon}, \\ \boldsymbol{\sigma}(\mathbf{u}_{\varepsilon}) \mathbf{n} = \mathbf{g} & \text{on } \partial\Omega, \\ \boldsymbol{\sigma}(\mathbf{u}_{\varepsilon}) \mathbf{e}^{\perp} = 0 & \text{on } \gamma_{\varepsilon}, \end{cases} \quad (2.2)$$

where \mathbf{n} is the outward unit normal to $\partial\Omega$ and $\boldsymbol{\sigma}(\mathbf{u}_{\varepsilon})$ is the stress defined by

$$\boldsymbol{\sigma}(\mathbf{u}_{\varepsilon}) = \mathbb{C} \nabla^s \mathbf{u}_{\varepsilon} := \frac{1}{2} \mathbb{C} (\nabla \mathbf{u}_{\varepsilon} + \nabla \mathbf{u}_{\varepsilon}^T). \quad (2.3)$$

Here, $\nabla^s \mathbf{u}_{\varepsilon} = \frac{1}{2} (\nabla \mathbf{u}_{\varepsilon} + \nabla \mathbf{u}_{\varepsilon}^T)$ is the strain tensor and the superscript T denotes the transpose of a matrix. Note that the functions in Ψ are solutions to the homogeneous problem (2.2) with $\mathbf{g} = 0$. So we impose orthogonality condition on \mathbf{u}_{ε} to guarantee the uniqueness of a solution to (2.2):

$$\int_{\partial\Omega} \mathbf{u}_{\varepsilon} \cdot \boldsymbol{\psi} \, d\sigma = 0 \quad \text{for all } \boldsymbol{\psi} \in \Psi. \quad (2.4)$$

Let \mathbf{u}_0 be the solution in the absence of the crack, *i.e.*, the solution to

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_0) = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}_0) \mathbf{n} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \quad (2.5)$$

with the orthogonality condition: $\mathbf{u}_0|_{\partial\Omega} \in L_{\Psi}^2(\partial\Omega)$ (or equivalently, (2.4) with \mathbf{u}_{ε} replaced with \mathbf{u}_0).

It is well-known that the solution \mathbf{u}_{ε} to (2.2) belongs to $H^1(\Omega \setminus \gamma_{\varepsilon})$. In fact, we have

$$\|\mathbf{u}_{\varepsilon}\|_{H^1(\Omega \setminus \gamma_{\varepsilon})} \leq C \quad (2.6)$$

for some C independent of ε . To see this, we introduce the potential energy functional

$$J_{\varepsilon}[\mathbf{u}] := -\frac{1}{2} \int_{\Omega \setminus \gamma_{\varepsilon}} \boldsymbol{\sigma}(\mathbf{u}) : \nabla^s \mathbf{u}. \quad (2.7)$$

The solution \mathbf{u}_{ε} of (2.2) is the maximizer of J_{ε} , *i.e.*,

$$J_{\varepsilon}[\mathbf{u}_{\varepsilon}] = \max J_{\varepsilon}[\mathbf{u}], \quad (2.8)$$

where the maximum is taken over all $\mathbf{u} \in H^1(\Omega \setminus \gamma_{\varepsilon})$ satisfying $\boldsymbol{\sigma}(\mathbf{u}) \mathbf{n} = \mathbf{g}$ on $\partial\Omega$ and $\boldsymbol{\sigma}(\mathbf{u}) \mathbf{e}^{\perp} = 0$ on γ_{ε} . Let \mathbf{v} be a smooth function with a compact support in Ω such that $\boldsymbol{\sigma}(\mathbf{v}) \mathbf{e}^{\perp} = -\boldsymbol{\sigma}(\mathbf{u}_0) \mathbf{e}^{\perp}$ on γ_{ε} . We may choose \mathbf{v} so that $J_{\varepsilon}[\mathbf{v}]$ is independent of ε . Since $0 \geq J_{\varepsilon}[\mathbf{u}_{\varepsilon}] \geq J_{\varepsilon}[\mathbf{u}_0 + \mathbf{v}]$, we have

$$\|\nabla^s \mathbf{u}_{\varepsilon}\|_{L^2(\Omega \setminus \gamma_{\varepsilon})} \leq C.$$

We then have from the Korn's inequality that there is a constant C independent of ε such that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^1(\Omega \setminus \gamma_\varepsilon)} \leq C(\|\nabla^s(\mathbf{u}_\varepsilon - \mathbf{u}_0)\|_{L^2(\Omega \setminus \gamma_\varepsilon)} + \|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^{1/2}(\partial\Omega)}). \quad (2.9)$$

Since $\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{H^{1/2}(\partial\Omega)}$ is bounded regardless of ε as we shall show later (Theorem 4.1), we obtain (2.6).

Let

$$\varphi_\varepsilon(\mathbf{x}) := \mathbf{u}_\varepsilon|_+(\mathbf{x}) - \mathbf{u}_\varepsilon|_-(\mathbf{x}), \quad \mathbf{x} \in \gamma_\varepsilon, \quad (2.10)$$

where $+$ (resp. $-$) indicates the limit on the crack γ_ε from the given normal direction \mathbf{e}^\perp (resp. opposite direction), *i.e.*,

$$\mathbf{u}_\pm(x) := \lim_{t \rightarrow 0} \mathbf{u}(\mathbf{x} \pm t\mathbf{e}^\perp).$$

We sometimes denote $\sigma(\mathbf{u})\mathbf{n}$, the traction on $\partial\Omega$ (or on γ_ε), by $\partial\mathbf{u}/\partial\nu$, *i.e.*,

$$\frac{\partial\mathbf{u}}{\partial\nu} := \lambda(\nabla \cdot \mathbf{u})\mathbf{n} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\mathbf{n} \quad \text{on } \partial\Omega. \quad (2.11)$$

If $\Phi = (\Phi_{ij})_{2 \times 2}$ is the Kelvin matrix of the fundamental solutions of Lamé system, *i.e.*,

$$\Phi_{ij}(\mathbf{x}) := \frac{A}{2\pi} \delta_{ij} \log |\mathbf{x}| - \frac{B}{2\pi} \frac{x_i x_j}{|\mathbf{x}|^2}, \quad \mathbf{x} \neq 0 \in \mathbb{R}^2, \quad (2.12)$$

where

$$A = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} \quad \text{and} \quad B = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}, \quad (2.13)$$

then the solution \mathbf{u}_ε to (2.2) is represented as

$$\begin{aligned} \mathbf{u}_\varepsilon(\mathbf{x}) = & \int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y})\mathbf{u}_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}) - \int_{\partial\Omega} \Phi(\mathbf{x} - \mathbf{y}) \frac{\partial\mathbf{u}_\varepsilon}{\partial\nu}(\mathbf{y}) d\sigma(\mathbf{y}) \\ & - \int_{\gamma_\varepsilon} \frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y})\varphi_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \Omega \setminus \gamma_\varepsilon. \end{aligned} \quad (2.14)$$

The solution \mathbf{u}_0 to (2.5) is represented as

$$\mathbf{u}_0(\mathbf{x}) = \int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y})\mathbf{u}_0(\mathbf{y}) d\sigma(\mathbf{y}) - \int_{\partial\Omega} \Phi(\mathbf{x} - \mathbf{y}) \frac{\partial\mathbf{u}_0}{\partial\nu}(\mathbf{y}) d\sigma(\mathbf{y}).$$

Let

$$\mathbf{w}_\varepsilon := \mathbf{u}_\varepsilon - \mathbf{u}_0. \quad (2.15)$$

Since $\frac{\partial\mathbf{u}_\varepsilon}{\partial\nu} = \frac{\partial\mathbf{u}_0}{\partial\nu}$ on $\partial\Omega$, by subtracting above two identities, we have

$$\mathbf{w}_\varepsilon(\mathbf{x}) - \mathcal{D}_\Omega[\mathbf{w}_\varepsilon](\mathbf{x}) = -\mathcal{D}_\varepsilon[\varphi_\varepsilon](\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (2.16)$$

where the double layer potentials \mathcal{D}_Ω and \mathcal{D}_ε are defined by

$$\mathcal{D}_\Omega[\mathbf{w}_\varepsilon](\mathbf{x}) := \int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y})\mathbf{w}_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \Omega \quad (2.17)$$

and

$$\mathcal{D}_\varepsilon[\varphi_\varepsilon](\mathbf{x}) := \int_{\gamma_\varepsilon} \frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \varphi_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \Omega. \quad (2.18)$$

Let $\mathbf{n}_{\mathbf{x}}$ denote the outward normal to $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$ and let

$$\mathcal{D}_\Omega[\varphi]|_-(\mathbf{x}) = \lim_{t \rightarrow 0+} \mathcal{D}_\Omega[\varphi](\mathbf{x} + t\mathbf{n}_{\mathbf{x}}).$$

Then, it is well known (see, for example, [6]) that

$$\mathcal{D}_\Omega[\varphi]|_-(\mathbf{x}) = \left(\frac{1}{2}I + \mathcal{K}_\Omega \right) [\varphi](\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \partial\Omega,$$

where \mathcal{K}_Ω is the boundary integral operator defined by

$$\mathcal{K}_\Omega[\varphi](\mathbf{x}) := \text{p.v.} \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \mathbf{w}_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega,$$

and I is the identity operator. Here, p.v. stands for the Cauchy principal value. It then follows from (2.16) that

$$\left(-\frac{1}{2}I + \mathcal{K}_\Omega \right) [\mathbf{w}_\varepsilon](\mathbf{x}) = \mathcal{D}_\varepsilon[\varphi_\varepsilon](\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \quad (2.19)$$

Since $-\frac{1}{2}I + \mathcal{K}_\Omega$ is invertible on $L_\Psi^2(\partial\Omega)$ (see, for instance, [13]), we have

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \int_{\gamma_\varepsilon} \frac{\partial}{\partial \nu_{\mathbf{y}}} \left(-\frac{1}{2}I + \mathcal{K}_\Omega \right)^{-1} [\Phi(\cdot - \mathbf{y})](\mathbf{x}) \varphi_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega. \quad (2.20)$$

Note that

$$\left(-\frac{1}{2}I + \mathcal{K}_\Omega \right)^{-1} [\Phi(\cdot - \mathbf{y})](\mathbf{x}) = \mathbf{N}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \partial\Omega, \quad \mathbf{y} \in \Omega, \quad (2.21)$$

modulo a function in Ψ , where $\mathbf{N}(\mathbf{x}, \mathbf{y})$ is the Neumann function for the Lamé system on Ω , namely, for $\mathbf{y} \in \Omega$, $\mathbf{N}(\mathbf{x}, \mathbf{y})$ is the solution to

$$\begin{cases} \nabla \cdot \sigma(\mathbf{N}(\cdot, \mathbf{y})) = -\delta_{\mathbf{y}} \mathbf{I} & \text{in } \Omega, \\ \sigma(\mathbf{N}(\cdot, \mathbf{y})) \mathbf{n} = -\frac{1}{|\partial\Omega|} \mathbf{I} & \text{on } \partial\Omega, \end{cases} \quad (2.22)$$

subject to the orthogonality condition:

$$\int_{\partial\Omega} \mathbf{N}(\mathbf{x}, \mathbf{y}) \cdot \psi(\mathbf{x}) d\sigma(\mathbf{x}) = 0 \quad \text{for all } \psi \in \Psi.$$

Here, \mathbf{I} is the 2 identity matrix. See [6, 7, 17] for properties of the Neumann function and a proof of (2.21). Thus we obtain from (2.20) that

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \int_{\gamma_\varepsilon} \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, \mathbf{y}) \varphi_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega. \quad (2.23)$$

We now describe the scheme to derive an asymptotic expansion of $\mathbf{u}_\varepsilon - \mathbf{u}_0$ on $\partial\Omega$. Since $\frac{\partial \mathbf{u}_\varepsilon}{\partial \nu} = \sigma(\mathbf{u}_\varepsilon) \mathbf{e}^\perp = 0$ on γ_ε , we use (2.16) to obtain

$$\frac{\partial \mathbf{u}_0}{\partial \nu} + \frac{\partial}{\partial \nu} \mathcal{D}_\Omega[\mathbf{w}_\varepsilon] = \frac{\partial}{\partial \nu} \mathcal{D}_\varepsilon[\varphi_\varepsilon] \quad \text{on } \gamma_\varepsilon. \quad (2.24)$$

We solve this integral equation for φ_ε and then substitute it into (2.23) to derive an asymptotic expansion of \mathbf{u}_ε as $\varepsilon \rightarrow 0$.

3 Derivation of an explicit integral equation

In view of (2.24), we need to compute $\frac{\partial}{\partial \nu_{\mathbf{x}}} \left(\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \right)$ on γ_ε . As before \mathbf{e}^\perp is the unit normal to γ_ε , and denoted by $\mathbf{e}^\perp = (n_1, n_2)$. It is worth mentioning that n_1 and n_2 are constant since γ_ε is a line segment. It is convenient to use the following expression of the conormal derivative:

$$\frac{\partial \mathbf{u}}{\partial \nu} = T(\partial) \mathbf{u}, \quad (3.1)$$

where the operator $T(\partial) = T(\partial_1, \partial_2)$, where $\partial_j = \frac{\partial}{\partial x_j}$, is defined by

$$T(\xi_1, \xi_2) := \begin{bmatrix} (\lambda + 2\mu)n_1\xi_1 + \mu n_2\xi_2 & \mu n_2\xi_1 + \lambda n_1\xi_2 \\ \lambda n_2\xi_1 + \mu n_1\xi_2 & \mu n_1\xi_1 + (\lambda + 2\mu)n_2\xi_2 \end{bmatrix}. \quad (3.2)$$

We first obtain the following formula whose derivation will be given in Appendix A. For $\mathbf{x} \neq \mathbf{y}$, we have

$$\begin{aligned} \left(\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \right)_{ij} &= \left[a \delta_{ij} + b \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^2} \right] \sum_{l=1}^2 \frac{n_l(x_l - y_l)}{|\mathbf{x} - \mathbf{y}|^2} \\ &\quad - a \frac{n_j(x_i - y_i) - n_i(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^2}, \end{aligned} \quad (3.3)$$

where

$$a = -\frac{\mu}{2\pi(\lambda + 2\mu)}, \quad b = -\frac{(\lambda + \mu)}{\pi(\lambda + 2\mu)}. \quad (3.4)$$

Let $v_{ij} = \left(\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \right)_{ij}$ for convenience and let

$$W(\mathbf{x} - \mathbf{y}) := \frac{\partial}{\partial \nu_{\mathbf{x}}} \left(\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \right). \quad (3.5)$$

Then one can use (3.2) to derive

$$\begin{aligned} W(\mathbf{x} - \mathbf{y})_{11} &= n_1[(\lambda + 2\mu)\partial_1 v_{11} + \lambda \partial_2 v_{21}] + n_2(\mu \partial_2 v_{11} + \mu \partial_1 v_{21}), \\ W(\mathbf{x} - \mathbf{y})_{12} &= n_1[(\lambda + 2\mu)\partial_1 v_{12} + \lambda \partial_{x_2} v_{22}] + n_2(\mu \partial_2 v_{12} + \mu \partial_1 v_{22}), \\ W(\mathbf{x} - \mathbf{y})_{21} &= n_1(\mu \partial_2 v_{11} + \mu \partial_1 v_{21}) + n_2[\lambda \partial_1 v_{11} + (\lambda + 2\mu)\partial_2 v_{21}], \\ W(\mathbf{x} - \mathbf{y})_{22} &= n_1(\mu \partial_2 v_{12} + \mu \partial_1 v_{22}) + n_2[\lambda \partial_1 v_{12} + (\lambda + 2\mu)\partial_2 v_{22}]. \end{aligned}$$

Since the crack which we consider is a line segment with length ε in the domain $\Omega \subset \mathbb{R}^2$, we may assume, after rotation and translation if necessary, that it is given by

$$\gamma_\varepsilon = \{(x_1, 0) : -\varepsilon/2 \leq x_1 \leq \varepsilon/2\}. \quad (3.6)$$

In this case, one can check that

$$\begin{aligned}
\partial_2 v_{11} &= \left(a + b \frac{(x_1 - y_1)^2}{|\mathbf{x} - \mathbf{y}|^2} \right) \frac{1}{|\mathbf{x} - \mathbf{y}|^2} = \frac{a + b}{(x_1 - y_1)^2}, \\
\partial_1 v_{21} &= a \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^2} - 2 \frac{x_1 - y_1}{|\mathbf{x} - \mathbf{y}|^2} \frac{x_1 - y_1}{|x_1 - y_1|^2} \right) = -\frac{a}{(x_1 - y_1)^2}, \\
\partial_1 v_{12} &= -a \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^2} - 2 \frac{x_1 - y_1}{|\mathbf{x} - \mathbf{y}|^2} \frac{x_1 - y_1}{|x_1 - y_1|^2} \right) = \frac{a}{(x_1 - y_1)^2}, \\
\partial_2 v_{22} &= \frac{a}{(x_1 - y_1)^2} \\
\partial_1 v_{11} &= \partial_2 v_{21} = \partial_2 v_{12} = \partial_1 v_{22} = 0.
\end{aligned}$$

Since $\mathbf{e}^\perp = (n_1, n_2) = (0, 1)$, we have

$$\begin{aligned}
W(\mathbf{x} - \mathbf{y})_{11} &= \mu \partial_2 v_{11} + \mu \partial_1 v_{21} \\
&= \mu(a + b) \frac{1}{(x_1 - y_1)^2} - \mu a \frac{1}{(x_1 - y_1)^2} = \frac{\mu b}{(x_1 - y_1)^2}, \\
W(\mathbf{x} - \mathbf{y})_{12} &= \mu \partial_2 v_{12} + \mu \partial_1 v_{22} = 0, \\
W(\mathbf{x} - \mathbf{y})_{21} &= \lambda \partial_1 v_{11} + (\lambda + 2\mu) \partial_2 v_{21} = 0, \\
W(\mathbf{x} - \mathbf{y})_{22} &= \lambda \partial_1 v_{12} + (\lambda + 2\mu) \partial_2 v_{22} \\
&= \lambda a \frac{1}{(x_1 - y_1)^2} + (\lambda + 2\mu) a \frac{1}{(x_1 - y_1)^2} = \frac{2(\lambda + \mu)a}{(x_1 - y_1)^2},
\end{aligned}$$

that is,

$$W(\mathbf{x} - \mathbf{y}) = \frac{1}{(x_1 - y_1)^2} \begin{bmatrix} -\frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} & 0 \\ 0 & -\frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \end{bmatrix}. \quad (3.7)$$

Note that

$$\frac{\mu(\lambda + \mu)}{\lambda + 2\mu} = \frac{E}{4} \quad (3.8)$$

where E is the Young's modulus in two dimensions. So, we have

$$W(\mathbf{x} - \mathbf{y}) = -\frac{E}{4\pi} \frac{1}{(x_1 - y_1)^2} \mathbf{I}. \quad (3.9)$$

So far we have shown that if γ_ε is given by (3.6), then

$$\frac{\partial}{\partial \nu_{\mathbf{x}}} \mathcal{D}_\varepsilon[\varphi_\varepsilon](\mathbf{x}) = -\frac{E}{4\pi} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{\varphi_\varepsilon(y)}{(x - y)^2} dy, \quad \mathbf{x} = (x, 0), \quad -\varepsilon/2 < x < \varepsilon/2. \quad (3.10)$$

Here the integral is hyper-singular and should be understood as a finite part in the sense of Hadamard, which will be defined in the next section. So the integral equation (2.24) becomes

$$\frac{1}{\pi} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{\varphi_\varepsilon(y)}{(x - y)^2} dy = -\frac{4}{E} \mathbf{f}(x), \quad -\varepsilon/2 < x < \varepsilon/2, \quad (3.11)$$

where

$$\mathbf{f}(x) = \frac{\partial \mathbf{u}_0}{\partial \nu}(x, 0) + \frac{\partial}{\partial \nu} \mathcal{D}_\Omega[\mathbf{w}_\varepsilon](x, 0). \quad (3.12)$$

Define

$$\mathbf{f}_\varepsilon(x) := \mathbf{f}\left(\frac{\varepsilon}{2}x\right), \quad (3.13)$$

and

$$\psi_\varepsilon(x) := \frac{2}{\varepsilon} \varphi_\varepsilon\left(\frac{\varepsilon}{2}x\right), \quad -1 < x < 1. \quad (3.14)$$

Then the scaled integral equation is

$$\frac{1}{\pi} \int_{-1}^1 \frac{\psi_\varepsilon(y)}{(x-y)^2} dy = -\frac{4}{E} \mathbf{f}_\varepsilon(x), \quad -1 < x < 1, \quad (3.15)$$

which we solve in the next section.

4 Asymptotic expansion

The integral in (3.15) is understood as a finite-part in the sense of Hadamard [19, 20]: for $\psi \in \mathcal{C}^{1,\alpha}(-1, 1)$ ($0 < \alpha \leq 1$)

$$\oint_{-1}^1 \frac{\psi(y)}{(x-y)^2} dy = \lim_{\delta \rightarrow 0} \left[\int_{-1}^{x-\delta} \frac{\psi(y)}{(x-y)^2} dy + \int_{x+\delta}^1 \frac{\psi(y)}{(x-y)^2} dy - \frac{2\psi(x)}{\delta} \right]. \quad (4.1)$$

Define

$$\mathcal{A}[\psi](x) := \frac{1}{\pi} \oint_{-1}^1 \frac{\psi(y)}{(x-y)^2} dy, \quad |x| < 1. \quad (4.2)$$

It is known ([14, 18]) that

$$\mathcal{A}[\psi](x) = -\frac{d}{dx} \mathcal{H}[\psi](x), \quad (4.3)$$

where \mathcal{H} is the (finite) Hilbert transform, *i.e.*,

$$\mathcal{H}[\psi](x) = \text{p.v.} \frac{1}{\pi} \int_{-1}^1 \frac{\psi(y)}{x-y} dy. \quad (4.4)$$

More properties of finite-part integrals and principal-value integrals can be found in [17, 19, 20, 21, 25, 27].

If $\psi(-1) = \psi(1) = 0$, we have from (4.3) that

$$\mathcal{A}[\psi](x) = -\mathcal{H}[\psi'](x). \quad (4.5)$$

Thus we can invert the operator \mathcal{A} using the properties of \mathcal{H} . The set \mathcal{Y} , given by

$$\mathcal{Y} = \left\{ \varphi : \int_{-1}^1 \sqrt{1-x^2} |\varphi(x)|^2 dx < +\infty \right\}, \quad (4.6)$$

is a Hilbert space with the norm

$$\|\varphi\|_{\mathcal{Y}} = \left(\int_{-1}^1 \sqrt{1-x^2} |\varphi(x)|^2 dx \right)^{1/2}.$$

It is well known (see, for example, [7, section 5.2]) that \mathcal{H} maps \mathcal{Y} onto itself and its null space is the one dimensional space generated by $1/\sqrt{1-x^2}$. Therefore, if we define

$$\mathcal{X} = \left\{ \psi \in \mathcal{C}^0([-1, 1]) : \psi' \in \mathcal{Y}, \psi(-1) = \psi(1) = 0 \right\}, \quad (4.7)$$

where ψ' is the distributional derivative of ψ , then $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y}$ is invertible. We note that \mathcal{X} is a Banach space with the norm

$$\|\psi\|_{\mathcal{X}} = \|\psi\|_{L^\infty} + \|\psi'\|_{\mathcal{Y}}.$$

Using the Hilbert inversion formula (see, for example, [7, section 5.2]), we can check

$$\mathcal{A}^{-1}[1](x) = -\sqrt{1-x^2}, \quad (4.8)$$

$$\mathcal{A}^{-1}[y](x) = -\frac{x}{2}\sqrt{1-x^2}. \quad (4.9)$$

The equation (3.15) can be written as

$$\mathcal{A}[\psi_\varepsilon](x) = -\frac{4}{E}\mathbf{f}_\varepsilon(x), \quad -1 < x < 1. \quad (4.10)$$

The Taylor expansion yields

$$\frac{\partial \mathbf{u}_0}{\partial \nu}(\frac{\varepsilon}{2}x, 0) = \frac{\partial \mathbf{u}_0}{\partial \nu}(0) + \frac{\varepsilon x}{2} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \nu}(0) + e_1(x), \quad (4.11)$$

where $\partial/\partial t$ denotes the tangential derivative on γ_ε . The remainder term e_1 satisfies

$$|e_1(x)| \leq C\varepsilon^2|x|^2,$$

and in particular,

$$\|e_1\|_{\mathcal{Y}} \leq C\varepsilon^2. \quad (4.12)$$

On the other hand, since

$$\frac{\partial}{\partial \nu} \mathcal{D}_\Omega[\mathbf{w}_\varepsilon](\frac{\varepsilon}{2}x, 0) = \int_{\partial\Omega} \frac{\partial^2 \mathbf{N}}{\partial \nu_{\mathbf{x}} \partial \nu_{\mathbf{y}}}((\frac{\varepsilon}{2}x, 0), \mathbf{y}) \mathbf{w}_\varepsilon(\mathbf{y}) d\sigma(\mathbf{y}),$$

and γ_ε is away from $\partial\Omega$, one can see that

$$\left\| \frac{\partial}{\partial \nu} \mathcal{D}_\Omega[\mathbf{w}_\varepsilon](\frac{\varepsilon}{2}\cdot, 0) \right\|_{\mathcal{Y}} \leq C\|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)}. \quad (4.13)$$

Therefore, we have

$$\mathbf{f}_\varepsilon(x) = \frac{\partial \mathbf{u}_0}{\partial \nu}(0) + \frac{\varepsilon x}{2} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \nu}(0) + e(x), \quad (4.14)$$

where e satisfies

$$\|e\|_{\mathcal{Y}} \leq C\left(\varepsilon^2 + \|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)}\right). \quad (4.15)$$

We now obtain from (4.10) that

$$\psi_\varepsilon(x) = -\frac{4}{E} \left[\frac{\partial \mathbf{u}_0}{\partial \nu}(0) \mathcal{A}^{-1}[1](x) + \frac{\varepsilon}{2} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \nu}(0) \mathcal{A}^{-1}[y](x) + \mathcal{A}^{-1}[e](x) \right]. \quad (4.16)$$

Note that $\mathcal{E}_1(x) = \mathcal{A}^{-1}[e](x)$ satisfies

$$\|\mathcal{E}_1\|_{\mathcal{X}} \leq C\|e\|_{\mathcal{Y}} \leq C \left(\varepsilon^2 + \|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)} \right),$$

and in particular,

$$\|\mathcal{E}_1\|_{L^\infty(-1,1)} \leq C \left(\varepsilon^2 + \|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)} \right). \quad (4.17)$$

It then follows from (4.8) and (4.9) that

$$\psi_\varepsilon(x) = \frac{4}{E} \left[\frac{\partial \mathbf{u}_0}{\partial \nu}(0) \sqrt{1-x^2} + \frac{\varepsilon}{4} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \nu}(0) x \sqrt{1-x^2} + \mathcal{E}_1(x) \right]. \quad (4.18)$$

Thus we have from (3.14) that

$$\varphi_\varepsilon(x) = \frac{2}{E} \left[\frac{\partial \mathbf{u}_0}{\partial \nu}(0) \sqrt{\varepsilon^2 - 4x^2} + \frac{1}{2} \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \nu}(0) x \sqrt{\varepsilon^2 - 4x^2} + \mathcal{E}(x) \right], \quad (x, 0) \in \gamma_\varepsilon, \quad (4.19)$$

where $\mathcal{E}(x) = \frac{\varepsilon}{2} \mathcal{E}_1(\frac{2}{\varepsilon}x)$ satisfies

$$\|\mathcal{E}\|_{L^\infty(\gamma_\varepsilon)} \leq C\varepsilon \left(\varepsilon^2 + \|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)} \right). \quad (4.20)$$

Substituting (4.19) into (2.23) we obtain

$$\begin{aligned} \mathbf{w}_\varepsilon(\mathbf{x}) &= \frac{2}{E} \int_{\gamma_\varepsilon} \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, (y, 0)) \sqrt{\varepsilon^2 - 4y^2} dy \frac{\partial \mathbf{u}_0}{\partial \nu}(0) \\ &\quad + \frac{1}{E} \int_{\gamma_\varepsilon} \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, (y, 0)) y \sqrt{\varepsilon^2 - 4y^2} dy \frac{\partial^2 \mathbf{u}_0}{\partial t \partial \nu}(0) \\ &\quad + \frac{2}{E} \int_{\gamma_\varepsilon} \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, (y, 0)) \mathcal{E}(y) dy := I + II + III, \quad \mathbf{x} \in \partial\Omega. \end{aligned} \quad (4.21)$$

Since

$$\frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, (y, 0)) = \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) + \frac{\partial^2}{\partial t_{\mathbf{y}} \partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) y + O(y^2),$$

we have

$$\begin{aligned} \int_{\gamma_\varepsilon} \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, (y, 0)) \sqrt{\varepsilon^2 - 4y^2} dy &= \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) \int_{\gamma_\varepsilon} \sqrt{\varepsilon^2 - 4y^2} dy \\ &\quad + \frac{\partial^2}{\partial t_{\mathbf{y}} \partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) \int_{\gamma_\varepsilon} y \sqrt{\varepsilon^2 - 4y^2} dy + O(\varepsilon^4) \\ &= \frac{\pi \varepsilon^2}{2} \frac{\partial}{\partial \nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) + O(\varepsilon^4), \end{aligned}$$

and hence

$$I = \frac{\pi\varepsilon^2}{E} \frac{\partial}{\partial\nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) \frac{\partial \mathbf{u}_0}{\partial\nu}(0) + O(\varepsilon^4). \quad (4.22)$$

Here and throughout this paper, $O(\varepsilon^4)$ is in the sense of the uniform norm on $\partial\Omega$. Similarly, one can show that

$$II = O(\varepsilon^4). \quad (4.23)$$

So we obtain that

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \frac{\pi\varepsilon^2}{E} \frac{\partial}{\partial\nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) \frac{\partial \mathbf{u}_0}{\partial\nu}(0) + O(\varepsilon^4) + III. \quad (4.24)$$

In particular, we have

$$\|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C(\varepsilon^2 + |III|).$$

But, because of (4.20), we arrive at

$$|III| \leq C\varepsilon \|\mathcal{E}\|_{L^\infty(\gamma_\varepsilon)} \leq C\varepsilon^2(\varepsilon^2 + \|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)}),$$

and hence

$$\|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C\varepsilon^2(1 + \|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)}).$$

So, if ε is small enough, then

$$\|\mathbf{w}_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C\varepsilon^2. \quad (4.25)$$

It then follows from (4.24) that

$$\mathbf{w}_\varepsilon(\mathbf{x}) = \frac{\pi\varepsilon^2}{E} \frac{\partial}{\partial\nu_{\mathbf{y}}} \mathbf{N}(\mathbf{x}, 0) \frac{\partial \mathbf{u}_0}{\partial\nu}(0) + O(\varepsilon^4). \quad (4.26)$$

We obtain the following theorem.

Theorem 4.1. *Suppose that γ_ε is a linear crack of size ε and \mathbf{z} is the center of γ_ε . Then the solution to (2.2) has the following asymptotic expansion:*

$$(\mathbf{u}_\varepsilon - \mathbf{u}_0)(\mathbf{x}) = \frac{\pi\varepsilon^2}{E} \frac{\partial \mathbf{N}}{\partial\nu_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{z}} \frac{\partial \mathbf{u}_0}{\partial\nu}(\mathbf{z}) + O(\varepsilon^4) \quad (4.27)$$

uniformly on $\mathbf{x} \in \partial\Omega$. Here E is the Young's modulus.

It is worth emphasizing that in (4.27) the error is $O(\varepsilon^4)$ and the ε^3 -term vanishes. One can see from the derivation of (4.27) that the ε^3 -term vanishes because γ_ε is a line segment. If it is a curve, then we expect that the ε^3 -term does not vanish. We also emphasize that (4.27) is a point-wise asymptotic formula, and it can be used to design algorithms to reconstruct cracks from boundary measurements. We can also integrate this formula against the traction \mathbf{g} to obtain the asymptotic formula for the perturbation of the elastic energy as we do in the next section.

Similarly, if we consider the Dirichlet problem

$$\begin{cases} \nabla \cdot \sigma(\mathbf{u}_\varepsilon) = 0 & \text{in } \Omega \setminus \bar{\gamma}_\varepsilon, \\ \mathbf{u}_\varepsilon = \mathbf{f} & \text{on } \partial\Omega, \\ \sigma(\mathbf{u}_\varepsilon) \mathbf{e}^\perp = 0 & \text{on } \gamma_\varepsilon, \end{cases} \quad (4.28)$$

and denote the Green function of Lamé system in Ω by \mathbf{G} , then we get the following asymptotic expansion of its solution \mathbf{u}_ε .

Theorem 4.2. *Suppose that γ_ε is a linear crack of size ε , located at \mathbf{z} . Then the solution to (4.28) has the following asymptotic expansion:*

$$\frac{\partial}{\partial \nu}(\mathbf{u}_\varepsilon - \mathbf{u}_0)(\mathbf{x}) = \frac{\pi \varepsilon^2}{E} \frac{\partial^2 \mathbf{G}}{\partial \nu_{\mathbf{x}} \partial \nu_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{z}} \frac{\partial \mathbf{u}_0}{\partial \nu}(\mathbf{z}) + O(\varepsilon^4) \quad (4.29)$$

uniformly on $\mathbf{x} \in \partial\Omega$.

5 Topological derivative of the potential energy

The elastic potential energy functional of the cracked body is given by (2.7), while without the crack the energy functional is given by

$$J[\mathbf{u}_0] = -\frac{1}{2} \int_{\Omega} \sigma(\mathbf{u}_0) : \nabla^s \mathbf{u}_0. \quad (5.1)$$

By the divergence theorem we have

$$J_\varepsilon[\mathbf{u}_\varepsilon] - J[\mathbf{u}_0] = -\frac{1}{2} \int_{\partial\Omega} (\mathbf{u}_\varepsilon - \mathbf{u}_0) \cdot \mathbf{g} \, d\sigma. \quad (5.2)$$

Thus we obtain from (4.27)

$$J_\varepsilon[\mathbf{u}_\varepsilon] - J[\mathbf{u}_0] = -\frac{\pi \varepsilon^2}{2E} \frac{\partial \mathbf{u}_0}{\partial \nu}(\mathbf{z}) \frac{\partial}{\partial \nu_{\mathbf{y}}} \int_{\partial\Omega} \mathbf{N}(\mathbf{x}, \mathbf{y}) \mathbf{g}(\mathbf{x}) \, d\sigma(\mathbf{x}) \Big|_{\mathbf{y}=\mathbf{z}} + O(\varepsilon^4).$$

Since

$$\mathbf{u}_0(\mathbf{y}) = \int_{\partial\Omega} \mathbf{N}(\mathbf{x}, \mathbf{y}) \mathbf{g}(\mathbf{x}) \, d\sigma(\mathbf{x}), \quad \mathbf{y} \in \Omega,$$

we have

$$J_\varepsilon[\mathbf{u}_\varepsilon] - J[\mathbf{u}_0] = -\frac{\pi \varepsilon^2}{2E} \left| \frac{\partial \mathbf{u}_0}{\partial \nu}(\mathbf{z}) \right|^2 + O(\varepsilon^4). \quad (5.3)$$

We may write (5.3) in terms of the stress intensity factors. The (normalized) stress intensity factors K_I and K_{II} are defined by

$$K_I(\mathbf{u}_0, \mathbf{e}) := \sigma(\mathbf{u}_0) \mathbf{e}^\perp \cdot \mathbf{e}^\perp \quad \text{and} \quad K_{II}(\mathbf{u}_0, \mathbf{e}) := \sigma(\mathbf{u}_0) \mathbf{e}^\perp \cdot \mathbf{e}. \quad (5.4)$$

So, we have

$$\sigma(\mathbf{u}_0) \mathbf{e}^\perp = K_I \mathbf{e}^\perp + K_{II} \mathbf{e}, \quad (5.5)$$

and hence

$$\left| \frac{\partial \mathbf{u}_0}{\partial \nu}(\mathbf{z}) \right|^2 = |\sigma(\mathbf{u}_0) \mathbf{e}^\perp|^2 = K_I^2 + K_{II}^2. \quad (5.6)$$

We obtain the following result.

Theorem 5.1. *We have*

$$J_\varepsilon[\mathbf{u}_\varepsilon] - J[\mathbf{u}_0] = -\frac{\pi \varepsilon^2}{2E} (K_I^2 + K_{II}^2) + O(\varepsilon^4) \quad (5.7)$$

as $\varepsilon \rightarrow 0$.

The topological derivative $D_T J_\varepsilon(\mathbf{z})$ of the potential energy is defined by [23, 24]

$$D_T J_\varepsilon(\mathbf{z}) := \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\rho'(\varepsilon)} \frac{d}{d\varepsilon} J_\varepsilon \right), \quad (5.8)$$

where $\rho(\varepsilon) = \pi\varepsilon^2$. So, one can immediately see from (5.7) that

$$D_T J_\varepsilon(\mathbf{z}) = -\frac{1}{2E}(K_I^2 + K_{II}^2). \quad (5.9)$$

This formula is in accordance with the one obtained by Novotny *et al* in [22] (see also [26]). In fact, in those papers the plane strain and the plain stress problems are considered, and (5.9) is the formula for the latter problem.

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A Derivation of (3.3)

The Kelvin matrix (2.12) can be rewritten as

$$\Phi_{ij}(\mathbf{x} - \mathbf{y}) = \lambda' \delta_{ij} \log |\mathbf{x} - \mathbf{y}| + \mu' (x_i - y_i) \frac{\partial \log |\mathbf{x} - \mathbf{y}|}{\partial y_j}, \quad i, j = 1, 2, \quad (A.1)$$

where

$$\lambda' = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}, \quad \mu' = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)}.$$

Using the operator $T(\partial)$ defined by (3.2) one can see that

$$\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) = (T(\partial_{\mathbf{y}}) \Phi(\mathbf{x} - \mathbf{y}))^T, \quad (A.2)$$

or

$$\left(\frac{\partial \Phi}{\partial \nu_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) \right)_{kj} := \sum_{l=1}^2 T_{jl}(\partial_{\mathbf{y}}) \Phi_{lk}(\mathbf{x} - \mathbf{y}), \quad k, j = 1, 2. \quad (A.3)$$

We use the formulas

$$\begin{aligned} \frac{\partial}{\partial y_i} \log |\mathbf{x} - \mathbf{y}| &= -\frac{x_i - y_i}{|\mathbf{x} - \mathbf{y}|^2}, \\ \frac{\partial^2}{\partial y_i^2} \log |\mathbf{x} - \mathbf{y}| &= -2 \frac{(x_i - y_i)^2}{|\mathbf{x} - \mathbf{y}|^4} + \frac{1}{|\mathbf{x} - \mathbf{y}|^2}, \\ \frac{\partial^2}{\partial y_i \partial y_j} \log |\mathbf{x} - \mathbf{y}| &= -2 \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^4} \quad \text{if } i \neq j. \end{aligned}$$

By (A.3), we have

$$\begin{aligned}
\left(\frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}\right)_{11} &= \left((\lambda+2\mu)n_1\frac{\partial}{\partial y_1} + \mu n_2\frac{\partial}{\partial y_2}\right) \left(\lambda' \log|\mathbf{x}-\mathbf{y}| + \mu'(x_1-y_1)\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_1}\right) \\
&\quad + \left(\mu n_2\frac{\partial}{\partial y_1} + \lambda n_1\frac{\partial}{\partial y_2}\right) \mu'(x_1-y_1)\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_2} \\
&= \lambda'(\lambda+2\mu)n_1\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_1} + \lambda'\mu n_2\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_2} \\
&\quad + \mu'(\lambda+2\mu)n_1 \left(-\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_1} + (x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_1^2}\right) \\
&\quad - \mu\mu' n_2\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_2} + 2\mu\mu' n_2(x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_1\partial y_2} \\
&\quad + \lambda\mu'(x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_2^2}.
\end{aligned}$$

Since $\Delta \log|\mathbf{x}-\mathbf{y}| = 0$ for $\mathbf{x} \neq \mathbf{y}$, we have

$$\begin{aligned}
\left(\frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}\right)_{11} &= (\lambda+2\mu)(\lambda'-\mu')n_1\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_1} + \mu(\lambda'-\mu')n_2\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_2} \\
&\quad + 2\mu\mu' \left(n_1(x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_1^2} + n_2(x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_1\partial y_2}\right).
\end{aligned}$$

Since

$$(\lambda+2\mu)(\mu'-\lambda') + 2\mu\mu' = -\frac{\mu}{2\pi(\lambda+2\mu)} = \mu(\mu'-\lambda'),$$

we obtain

$$\left(\frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}\right)_{11} = \left[\mu(\mu'-\lambda') - 4\mu\mu'\frac{(x_1-y_1)^2}{|\mathbf{x}-\mathbf{y}|^2}\right] \sum_{l=1}^2 n_l \frac{x_l-y_l}{|\mathbf{x}-\mathbf{y}|^2}.$$

Similarly, we can compute

$$\begin{aligned}
\left(\frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}\right)_{12} &= \left(\lambda n_2\frac{\partial}{\partial y_1} + \mu n_1\frac{\partial}{\partial y_2}\right) \left(\lambda' \log|\mathbf{x}-\mathbf{y}| + \mu'(x_1-y_1)\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_1}\right) \\
&\quad + \left(\mu n_1\frac{\partial}{\partial y_1} + (\lambda+2\mu)n_2\frac{\partial}{\partial y_2}\right) \mu'(x_1-y_1)\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_2} \\
&= \lambda(\lambda'-\mu')n_2\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_1} + \mu(\lambda'-\mu')n_1\frac{\partial \log|\mathbf{x}-\mathbf{y}|}{\partial y_2} \\
&\quad + 2\mu\mu' n_1(x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_1\partial y_2} + 2\mu\mu' n_2(x_1-y_1)\frac{\partial^2 \log|\mathbf{x}-\mathbf{y}|}{\partial y_2^2}.
\end{aligned}$$

Since $\lambda(\mu' - \lambda') + 2\mu\mu' = \mu(\lambda' - \mu')$, we have

$$\begin{aligned}
\left(\frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}\right)_{12} &= [\lambda(\mu' - \lambda') + 2\mu\mu']n_2\frac{x_1 - y_1}{|\mathbf{x} - \mathbf{y}|^2} + \mu(\mu' - \lambda')n_1\frac{x_2 - y_2}{|\mathbf{x} - \mathbf{y}|^2} \\
&\quad - 4\mu\mu'n_1\frac{(x_1 - y_1)^2(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^4} - 4\mu\mu'n_2\frac{(x_1 - y_1)(x_2 - y_2)^2}{|\mathbf{x} - \mathbf{y}|^4} \\
&= -4\mu\mu'\frac{(x_1 - y_1)(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^2} \sum_{l=1}^2 n_l \frac{x_l - y_l}{|\mathbf{x} - \mathbf{y}|^2} \\
&\quad - \mu(\mu' - \lambda') \left[\frac{n_2(x_1 - y_1) - n_1(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^2} \right].
\end{aligned}$$

We also have

$$\begin{aligned}
\left(\frac{\partial\Phi}{\partial\nu_{\mathbf{y}}}\right)_{22} &= \left(\lambda n_2 \frac{\partial}{\partial y_1} + \mu n_1 \frac{\partial}{\partial y_2}\right) \mu'(x_2 - y_2) \frac{\partial \log |\mathbf{x} - \mathbf{y}|}{\partial y_1} \\
&\quad + \left(\mu n_1 \frac{\partial}{\partial y_1} + (\lambda + 2\mu)n_2 \frac{\partial}{\partial y_2}\right) \left(\lambda' \log |\mathbf{x} - \mathbf{y}| + \mu'(x_2 - y_2) \frac{\partial \log |\mathbf{x} - \mathbf{y}|}{\partial y_2}\right) \\
&= \left[\mu(\mu' - \lambda') - 4\mu\mu' \frac{(x_2 - y_2)^2}{|\mathbf{x} - \mathbf{y}|^2}\right] \sum_{l=1}^2 n_l \frac{x_l - y_l}{|\mathbf{x} - \mathbf{y}|^2}.
\end{aligned}$$

This proves (3.3).

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